$\boxed{\text{Lecture 25}}$

$\lambda \in \mathfrak{t}^*$ analytically integral weight $\left(\exp(h) \mapsto \underbrace{e^{i\lambda(h)}}_{\tilde{\lambda}(\exp(h))} \text{ is well-def}\right)$

$\mathbb{C}_\lambda = \mathbb{C}$ as B-module w/ $b.z = \tilde{\lambda}(b)^{-1} z$.

$$\begin{array}{ccc} \mathcal{L}_\lambda & = & (G \times \mathbb{C}_\lambda)/B \\ \downarrow \pi & & \downarrow \\ G/B & & G/B \end{array}$$

Section: $\sigma : G/B \to \mathcal{L}_\lambda$ s.t. $\pi \circ \sigma = id_{G/B}$

Equiv to $f : G \to \mathbb{C}_\lambda$ s.t. $f(gb) = b \cdot f(g)$
$\sigma(gB) = [\![g, f(g)]\!]$

<u>Defining $c_1(\mathcal{L}_\lambda) \in H^2(G/B, \mathbb{R})$</u>

① Euler class

$\mathcal{L}_\lambda$ may or may not have a holomorphic section!

(1a) It will have a <u>meromorphic</u> section. ($f : G \to \mathbb{C}_\lambda$ meromorphic)

Then just as mero fns on $\Omega \subset \mathbb{C}$ have

$$div(f) = \sum_{z \in \text{Zeros } f} (\text{order of zero}) \cdot z - \sum_{p \in \text{Poles } f} (\text{order of poles}) \cdot p$$

For higher dim cases like this, the divisor of $f : G \to \mathbb{C}_\lambda$ is an integer linear combination of irred proj subvar of codim 1.

$div(f) = \sum n_i \tilde{X}_i$ In a slice chart of a smooth point of $\tilde{X}_i$, $f(z_1, \dots, z_n)$ looks like $C z_1^{n_i}$ (holo nonvanish)

The $\tilde{X}_i$ are B-invt so descend to $X_i \subset G/B$

$$e(\mathcal{L}_\lambda) = \sum n_i [X_i] \in H^2(G/B, \mathbb{R}).$$

$$c_1(\mathcal{L}_\lambda) = e(\mathcal{L}_\lambda)$$

(1b) Construct a <u>smooth</u> section so intersections with zero section have $T_p\sigma \cap T_p\sigma_z$ of dim $2N-2$ everywhere.

Then $\sigma^{-1}(0) = Z_+ \cup Z_-$, orientation determines $+, -$

$e(\mathcal{L}_\lambda) = [Z_+] - [Z_-]$.

(1c) Restrict $\mathcal{L}_\lambda$ to the $\mathbb{CP}^1$ $[X_{\alpha_i}] \subset G/B$. degree

Get a holo line bundle on $\mathbb{CP}^1$, so it's $\mathcal{O}(d_i)$

$e(\mathcal{L}_\lambda)$ = the element of H^2 such that

$\langle e(\mathcal{L}_\lambda), [X_{\alpha_i}] \rangle = d_i$ for all i

(2) Universal constructions.

$\mathcal{L}_\lambda$ is a complex line bundle on G/B. It has a holo str, but ignore!

There is a universal line bundle $\mathbb{L} \to \mathbb{B}$ s.t.:

If $\mathcal{L} \to X$ is a continuous cplx line bundle, then $\exists$ cts map $F_{\mathcal{L}}: X \to \mathbb{B}$ s.t. $\mathcal{L} \cong F_{\mathcal{L}}^*(\mathbb{L})$, i.e.

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, "\pi"'] & \mathbb{L} \arrow[d, "\pi_{\mathbb{L}}"] \\ X \arrow[r, "F_{\mathcal{L}}"'] & \mathbb{B} \end{tikzcd}$$

is a pullback diagram.

What is this amazing object? Starts with ES^1 = A contractible CW complex with a free action of S^1. (EG similar!)

it's unique up to htpy equiv

Then $\mathbb{B} = BS^1 := ES^1/S^1$.

And $\mathbb{L} = (ES^1 \times \mathbb{C})/S^1$. where $S^1 \subset \mathbb{C}^*$ acts on $\mathbb{C}$ by mult
$U(1) \subset GL_1\mathbb{C}$

OK, what's ES^1? It's S^∞! (Yes, S^∞ is contractible...)

View S^∞ as living in $\mathbb{C}^\infty$. Then S^1 acts as mult by $(e^{i\theta}, e^{i\theta}, \ldots)$.

The quotient $BS^1 = \mathbb{B}$ is $\mathbb{CP}^\infty$.

Now $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x]$ w/ x in degree 2. We can take $x \in H^2(\mathbb{CP}^\infty) \cong \mathrm{Hom}(H_2(\mathbb{CP}^\infty), \mathbb{Z})$ to be the dual basis of the 2-skeleton of $\mathbb{CP}^\infty$ which is $\mathbb{CP}^1 \cong S^2$ (cplx str giving a fund class).

Now $c_1(\mathcal{L}) = F_{\mathcal{L}}^*(x)$.

Recap of where we are w/ Borel:

$$\underset{\text{coinvts}}{\mathcal{R}} \xrightarrow{\ c\ } H^*(G/B, \mathbb{R})$$

$c(\lambda)$ is just $c_1(\mathcal{L}_\lambda)$. Extends to a homomorphism. Why iso?
↳ integral in $\mathfrak{t}^* = \mathcal{R}_1$

I'll sketch an approach (not Borel's).

$EG =$ space w/ free G-action, contractible.

$$\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$$ principal G-bundle (universal)

For example, $\begin{tikzcd} G \arrow[d] \\ \bullet \end{tikzcd}$ is pullback:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[d] \\ \bullet \arrow[r] & BG \end{tikzcd}$$

Now take the quotient of the top line by B:

EG can be EB so $EG/B = BB$

$$\begin{tikzcd} G/B \arrow[r] \arrow[d] & BB \arrow[d] \\ \bullet \arrow[r] & BG \end{tikzcd}$$ (pullback)

Now take cohomology:

$$\begin{tikzcd} H^*(G/B) & H^*(BB) \arrow[l] \\ H^*(\bullet) \arrow[u] & H^*(BG) \arrow[l, "i"] \arrow[u, "j"'] \end{tikzcd}$$

j is injective ($BB \to BG$ fiber bdl)

Now i is determined, it's an iso $H^0(BG) \cong H^0(\cdot)$ and zero else.

What about j?

$$\begin{tikzcd} H^*(BB) \arrow[r, phantom, "\cong"] \arrow[r, bend right, "\text{B htpy equiv to } H"'] & H^*(BH) \arrow[r, phantom, "\cong"] \arrow[r, bend left, "H = (\mathbb{C}^*)^n \text{ htpy to } (S^1)^n"] & H^*(B(S^1)^n) = H^*((\mathbb{CP}^\infty)^r) \cong \mathbb{Z}[x_1, \dots, x_r] \end{tikzcd}$$

W acts on BB hence on $H^*(BB)$.

W also acts on BG and $H^*(BG)$ but there it acts trivially since G is conn and $w \in W$ is repr by elt of G.

$$\text{So} \quad \begin{tikzcd} H^*(BG) \arrow[r, "j"] \arrow[loop below, "W \text{ trivial}"] & H^*(BB) \arrow[loop below, "W\ ?"] \end{tikzcd} \quad \text{image must be in } H^*(BB)^W = \mathbb{Z}[x_1 \dots x_r]^W.$$

<u>Fact</u>. The image of j is exactly $H^*(BB)^W$.

Finally:
$$\begin{tikzcd} H^*(G/B) & \mathcal{P} \arrow[l] \\ {[\mathbb{Z}]_0} \arrow[u] & \mathcal{P}^W \arrow[l] \arrow[u] \end{tikzcd}$$

Now, you might think the orig diag being a pullback would mean that this diag is a pushout, i.e.

$$H^*(G/B) \overset{?}{=} \mathcal{P} \otimes_{\mathcal{P}^W} [\mathbb{Z}]_0 \qquad (\text{tensor prod of } \mathcal{P}^W\text{-mod})$$

But in general that doesn't work, as $\otimes$ not exact.

<u>Key</u>: $\mathcal{P}$ is a <u>free</u> module over $\mathcal{P}^W$ (Chevalley).

So $H^*(G/B) = P \otimes_{P^W} [\mathbb{Z}]_0 = P /$ ideal gen by $\ker(P^W \to [\mathbb{Z}]_0)$

$$= P / (P^W_+)$$